# Linear hydrodynamic model of rotating lift-based wave energy converters

Matt Folley, Paul Lamont-Kane, Carwyn Frost

*Abstract*— **A linear potential flow model of a rotating lift-based wave energy converter is developed by assuming that the lift is generated by a pair of equal and opposite circulations and that the amplitude of motion is small. The linearisation of the hydrodynamics means that the forces can decomposed and expressions for the wave excitation force and radiation damping force are derived independently and shown to be related to each other through the Haskind Relations. The expressions for the forces are used to show that there is an optimum phase and product of circulation and radius of rotation to maximise the wave power extracted, which is equivalent to the optimum phase and amplitude of motion from 'conventional' wave energy converter theory. It is also shown that at this optimum condition 100% of the incident wave energy can be extracted. It is shown that the forces are directly proportional to the velocities due to the motion of the vortices, the water particle velocities due to the incident wave, and the water particle velocities induced by the vortices. The effect of the vortex-induced water particle velocities is considered and the importance of including these velocities on the passive generation of circulation, e.g. by hydrofoils, is highlighted. The impact of a sub-optimum product of circulation and radius of rotation is also investigated and shown that the power capture is not highly sensitive to the optimal conditions in the same way as 'conventional' wave energy converters.**

*Keywords*— **circulation, design, hydrodynamics, hydrofoil, lift, wave energy**

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## I. INTRODUCTION

Wave energy remains one of the largest commercially untapped sources of renewable energy despite over 50 years of effort in its development during the modern era following the first oil crisis in the 1970s. In these 50 years, thousands of wave energy converters have been proposed for the extraction of wave energy but, unfortunately, none of these technologies has yet (as of 2022) shown that they are commercially viable. This is undoubtedly partly due to the complexity of the challenge (extracting energy from a highly variable source in an extremely aggressive environment) and the absence of historical prototypes, which technologies like wind turbines had with windmills, but it is also likely to be due to the vast range of potential methods of extracting energy from the waves. Of these different methods of extracting energy from the waves, the use of wave-induced lift forces has attracted relatively little attention, with the focus being on technologies such as using oscillating water columns (OWCs), heaving buoys and oscillating flaps (OWSCs) which use buoyancy and diffraction forces [1], [2][2].

The lack of interest in lift-based wave energy converter may be surprising as the generation of lift has been shown to be the most efficient method of extracting wind energy [3], albeit there is significant difference between the predominantly rectilinear velocity of the air due to wind and the circular motion of the water due to waves. Furthermore, the use of lift means that survival in storms is much less demanding as the force previously required to extract energy can be effectively removed by stopping

[2] It may be argued that OWCs are not excited by buoyancy and/or diffraction forces. However, in potential flow modelling the OWC water surface are generally represented as a mass-less piston located on the surface of the water column. The motion of this mass-less piston is excited by diffraction and buoyancy forces and its motion is then used to represent the motion of the water surface. Thus, at least mathematically an OWC is excited by buoyancy and diffraction forces and can be treated the same as other wave energy converters.

the generation of lift. Another peculiarity in the development of wave energy converters is that most current technologies extract power from a reciprocating motion, whilst the wave-induced motion of water particles are essentially circular (or elliptical in shallow water). This significantly complicates the extraction of energy from the waves as the power-take-off technology typically needs to cope with reversing loads and periods of zero velocity. There are likely to be multiple reasons for these focuses; however, a consequence of this is that there is limited fundamental understanding of how rotating lift-based wave energy converters can extract wave energy.

The first developments in the fundamental understanding of wave energy converters were made in the 1970s. It was shown that in two dimensions a symmetric wave energy converter could only capture a maximum of 50% of the incident wave energy, whilst a body moving in two or more modes could extract 100% of the energy [4]. Further analysis showed that the maximum wave energy capture requires an optimum amplitude of body motion and that the body velocity is in-phase with the wave excitation force [5]. Although further refinements of these analyses occurred in the 1980s with extension to three-dimensional bodies and consideration of such factors as motion constraints [6], these fundamental findings remain valid and a key tool in understanding how to optimise the design of a wave energy converter.

Importantly, these fundamental relationships are based on linear potential flow theory, and it is worth noting that although many higher fidelity models have subsequently been developed that enable a more accurate representation of the non-linear hydrodynamics, the understanding of wave energy converters design is still primarily based on these simplified linear hydrodynamic models (with in some cases perturbations to account for non-linearities) [1], [7], [8]. The advantage of these linear hydrodynamic models is not that they are more accurate, indeed in some cases they can be very inaccurate in terms of calculation of magnitude of response or power capture, but that they can provide valuable insight into the hydrodynamics. Although the higher fidelity models may be more accurate in predicting actual performance, their sheer complexity means that their formulations and results are more difficult to interpret.

Unfortunately, the fundamental understanding of wave energy converters that has been developed does not appear to be directly applicable to rotating lift-based wave energy converters. In its place, rotating lift-based wave energy converters has been largely developed using higher fidelity models [9], [10], which has enabled a more accurate estimate of the wave energy converter response and performance to be produced, but these in general provide limited insight into the optimum design. Thus, although it may be able to identify the optimum response to maximise power capture with the higher fidelity models, the reason for the higher power capture may not be clear. If the objective were simply to maximise power capture within clearly defined bounds of device characteristics this may be considered adequate; however, in the conceptual design this is generally not the case. Without a clear understanding of the fundamental hydrodynamics, the specified design space may be unknowingly constrained or distorted, which could result in sub-optimal solutions.

The value of linear hydrodynamics models to support the design of conventional wave energy converters suggests that developing a linear hydrodynamic model for rotating lift-based wave energy converters may have a similar value. The key with these simplified models is not that they provide an accurate estimate of response or power performance but provide the insight that supports the design process to interpret, explain and justify the optimal designs that may be produced by higher fidelity models.

Thus, the objective of this paper is the production of a linear hydrodynamic model of a lift-based wave energy converter that can be used to guide design. To achieve this, it is necessary that the model is as simple and fundamental as possible. This means that the paper is of the same family of paper that were produced in the 1970s on the fundamentals of more conventional wave energy converters that exploit buoyancy and/or diffraction forces. These models make minimal claim to provide a realistic estimate of the power production but do provide a powerful building block upon which more advanced models that can produce a more realistic estimate of power production can be developed. As such, the paper includes no consideration of model validation by comparison to physical model data. No apology is made for this omission as it would be misleading in the same way that comparison of the linear hydrodynamic models developed in the 1970s with wave energy converters in realistic conditions would have also illustrated their lack of ability to predict performance. It is considered that the value of these papers is not in their predictive ability but in their ability to provide insight into the hydrodynamics that can subsequently be used in design. Conversely, the objective of the paper is not to produce a model that is capable of predicting performance. However, it is anticipated that such a paper will be developed subsequently based on the fundamental understanding of the hydrodynamics developed in this paper, in the same way that the linear models of wave energy converters were built upon in subsequent papers.

This paper starts by formulating a linear potential flow solution for a two-dimensional rotating lift-based wave energy converter. This formulation is used to define equivalent optimum conditions for the maximisation of the extraction of wave energy for this type of wave energy converter, together with development of a fundamental understanding of its hydrodynamics. These results are then compared with those for a ‘conventional’ wave energy converter, identifying their similarities and differences. Finally, some consideration is given to the generation of circulation, which is core to a lift-based wave energy converter, and how this may affect the fundamental hydrodynamics and optimum design of this type of wave energy converter.

## II. FORMULATION

A linear potential flow solution is generated assuming that the lift-based wave energy converter can be represented as two point vortices with equal and opposite values of circulation $\pm\Gamma$ that are rotating with a 180° phase difference in the same direction as the wave-induced water particle motions and at an angular velocity equal to the wave frequency $\omega$. This rotation is about a fixed point at depth $h$ below the free-surface in an infinite water depth at a radius $c$ as shown in Fig. 1. All effects of viscosity are neglected, except for the requirement for the generation of circulation. A cartesian coordinate system $x, y$ is defined with its origin at the free surface directly above the centre of rotation of the two hydrofoils (see Fig. 1).

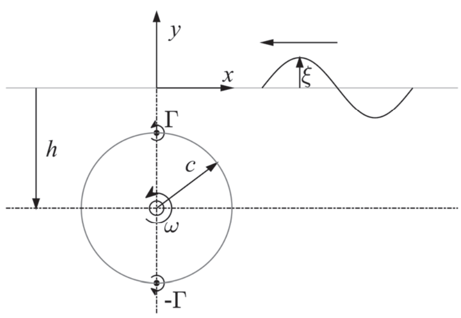


Fig. 1: Dimensions and coordinate system of rotating lift-based wave energy converter

By ensuring that the potential flow solution remains linear then it is possible to separate the forces into those due to the wave and those due to the motion of the body, with the total hydrodynamic force on the body simply the sum of these two forces. This is a standard approach in marine hydrodynamics, defining wave forces on the body in the absence of body motion and added damping and mass forces due to the motion of the body in the absence of waves [11].

First consider the complex velocity potential for the incident wave $w_i$, which is given for a wave amplitude $\xi$ and wave number $k$ as

$$w_i = -\mathrm{i}\frac{\xi\omega}{k}\mathrm{e}^{-\mathrm{i}(kz+\omega t+\gamma)} \quad (1)$$

where the complex potential is given by $w = \phi + \mathrm{i}\psi$ ($\phi$ is the velocity potential and $\psi$ is the stream function) and $z = x + \mathrm{i}y$ ($x$ is the horizontal coordinate and $y$ is the vertical coordinate positive upwards) and $\gamma$ is the phase difference between the wave crest and the rotational position of the positive circulation. The complex velocity induced by the wave $U_i$ can be calculated by differentiation with respect to z so that

$$U_i = -\omega\xi\mathrm{e}^{-\mathrm{i}(kz+\omega t+\gamma)} \quad (2)$$

The force on a vortex is given by the Kutta–Joukowski theorem as the cross product of a vortex's circulation and the incident velocity [12]. Assuming that the radius of rotation $c$ is small, which is equivalent to the infinitesimally small amplitude of motion used in linear potential flow solutions for conventional wave energy converters, then the wave-induced fluid velocity at the vortex will be the same as that at the centre of rotation.

Thus, the complex wave-induced force $F_i$ due to the vortex with the positive circulation is given by

$$F_i = -\rho\Gamma\omega\xi\mathrm{e}^{-kh}\mathrm{e}^{-\mathrm{i}(\omega t+\gamma)} \quad (3)$$

Equation (3) shows that the lift force is proportional to the wave amplitude and so linear as is the wave excitation force due to buoyancy (of a surface-piercing body) and diffraction, which are the wave forces used in the majority of conventional wave energy converters. The magnitude of these wave excitation forces cannot be compared directly since each is dependent on different characteristics of the body: lift – circulation, buoyancy – water plane area, diffraction – body shape. However, it is useful to compare the frequency dependence for these three types of wave excitation force since this can be used to identify different hydrodynamic characteristics as has been done previously for heaving and surging wave energy converters [13]. The frequency variation of the wave excitation force due to buoyancy, lift and diffraction is shown in Fig. 2, where the wave forces for lift and diffraction have been normalised by their maximum value and the wave frequency have been normalised by its value at the peak wave force.

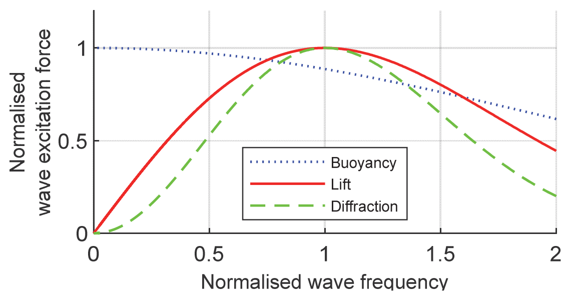


Fig. 2: Variation of normalized wave excitation force with normalized wave frequency

It can be seen that the wave excitation force due to lift has the same form of variation with wave frequency as for diffraction, except that it is less pronounced. This is because fundamentally the lift force is proportional to the wave-induced water particle velocity, which increases linearly with the wave frequency, whilst the diffraction force is proportional to the wave-induced water particle acceleration, which increases quadratically, and thus varies more, with the wave frequency. The buoyancy force decreases slowly with frequency due to the exponential decay in the Froude-Krylov force with depth below the water surface.

The net force from the two circulations can be calculated by using the wave-induced force given in (3) for positive and negative circulations. It can be easily seen that this results in a zero net force. However, in general the two forces are not aligned and so generate a torque about the centre of rotation. The torque is calculated using the cross product (×) of the force and distance of the circulation from the centre of rotation which is given by (4) as

$$T_i = -\rho\Gamma\omega\xi \mathrm{e}^{-kh}\mathrm{e}^{-\mathrm{i}(\omega t+\gamma)} \times c\mathrm{e}^{-\mathrm{i}\omega t} + \rho\Gamma\omega\xi \mathrm{e}^{-kh}\mathrm{e}^{-\mathrm{i}(\omega t+\gamma)} \times c\mathrm{e}^{-\mathrm{i}(\omega t+\pi)} \quad (4)$$

However, the cross-product of two exponentials is equal to the sine of their difference [14], so that (4) can be simplified as

$$T_i = -2\rho\Gamma c\omega\xi \mathrm{e}^{-kh}\sin\gamma \quad (5)$$

It is worth noting that this wave-induced excitation torque is time-invariant. Now consider the case of the vortices (circulations) rotating in still water. In an infinite expanse of water then the complex velocity potential $w_0$ at time $t$ for a single vortex with positive circulation is given by Basset [15] as

$$w_0 = -\mathrm{i}\frac{\Gamma}{2\pi}\ln(z + \mathrm{i}h - c\mathrm{e}^{-\mathrm{i}\omega t}) \quad (6)$$

Adding a second vortex with the opposite circulation and a 180° phase shift gives the complex velocity potential for the pair of vortices as

$$w_0 = -\mathrm{i}\frac{\Gamma}{2\pi}\ln\left(\frac{z + \mathrm{i}h - c\mathrm{e}^{-\mathrm{i}\omega t}}{z + \mathrm{i}h + c\mathrm{e}^{-\mathrm{i}\omega t}}\right) \quad (7)$$

To keep the solution linear, it is necessary to assume that the radius of rotation $c$ is small, which is equivalent to the infinitesimally small amplitude of motion used in linear potential flow solutions for 'conventional' wave energy converters [11]. In this case the linearized complex velocity potential becomes

$$w_0 = -\mathrm{i}\frac{\Gamma c}{\pi}\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{(z + \mathrm{i}h)} \quad (8)$$

To account for the effect of the free surface, a complex potential $w_s$ is required that has the same form in the vicinity of the axes of rotation $z = -\mathrm{i}h$ and also satisfies the linearised free surface boundary condition, which is

$$\Im m\left\{\frac{\mathrm{i}}{g}\frac{\partial^2 w_s}{\partial t^2} - \frac{\partial w_s}{\partial z}\right\} = 0 \text{ on } y = 0 \quad (9)$$

It should also have no other singularities in the lower half-plane $y < 0$. These requirements are met by the solution to the equation

$$\frac{\mathrm{i}}{g}\frac{\partial^2 w_s}{\partial t^2} - \frac{\partial w_s}{\partial z} = \frac{\mathrm{i}}{g}\frac{\partial^2 w_0}{\partial t^2} - \frac{\partial w_0}{\partial z} + G(z,t) \quad (10)$$

provided that the unknown function $G(z,t)$ also has no singularities in the lower half-plane.

Using the Schwarz reflection theorem [16] this can be satisfied when $G(z,t)$ is a circulation mirrored in the free surface, which is given by the complex velocity potential $\widetilde{w}_0$

$$\widetilde{w}_0 = \mathrm{i}\frac{\Gamma c}{\pi}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{(z - \mathrm{i}h)} \quad (11)$$

Thus, (10) becomes

$$\frac{\mathrm{i}}{g}\frac{\partial^2 w_s}{\partial t^2} - \frac{\partial w_s}{\partial z} = \frac{\mathrm{i}}{g}\frac{\partial^2 (w_0 + \widetilde{w}_0)}{\partial t^2} - \frac{\partial (w_0 + \widetilde{w}_0)}{\partial z} \quad (12)$$

Then, using simple differentiation and defining the wave number $k = \omega^2/g$ it can be shown that (12) becomes

$$\frac{\partial w_s}{\partial z} + \mathrm{i}kw_s = \frac{\Gamma c}{\pi}\left[\left(\frac{-k}{(z + \mathrm{i}h)} + \frac{\mathrm{i}}{(z + \mathrm{i}h)^2}\right)\mathrm{e}^{\mathrm{i}\omega t} + \left(\frac{k}{(z - \mathrm{i}h)} - \frac{\mathrm{i}}{(z - \mathrm{i}h)^2}\right)\mathrm{e}^{-\mathrm{i}\omega t}\right] \quad (13)$$

Equation (13) is a linear first-order non-homogeneous differential equation, so can be solved using standard techniques [17] and has the solution

$$w_s = \frac{\Gamma c}{\pi}\left[\left(-2k\mathrm{e}^{-\mathrm{i}k(z+\mathrm{i}h)}\mathrm{E_i}\big(-\mathrm{i}k(z + \mathrm{i}h)\big) - \frac{\mathrm{i}}{z + \mathrm{i}h} + C_1\right)\mathrm{e}^{\mathrm{i}\omega t} + \left(2ke^{-\mathrm{i}k(z-\mathrm{i}h)}\mathrm{E_i}\big(-\mathrm{i}k(z - \mathrm{i}h)\big) + \frac{\mathrm{i}}{z - \mathrm{i}h} + C_2\right)\mathrm{e}^{-\mathrm{i}\omega t}\right] \quad (14)$$

Where $C_1, C_2$ are constants of integration and $\mathrm{E_i}$ is the Exponential Integral given by

$$\mathrm{E_i}(x) = \int_x^\infty \frac{e^{-s}}{s}ds \quad (15)$$

The force due to the motion of the vortex can again be solved using the Kutta–Joukowski theorem and is equal to the cross-product of the circulation and circulation-induced velocity at the location of each vortex. Thus, to calculate the force it is necessary to determine the velocity induced by the circulation. This velocity $U_v$ can be calculated by differentiating the complex velocity potential $w_s$ with respect to $z$ as shown in (16)

$$U_v = \frac{\partial w_s}{\partial z} = \mathrm{i}\frac{\Gamma c}{2\pi}\Bigg[\Bigg(4k^2\mathrm{e}^{-\mathrm{i}k(z+\mathrm{i}h)}\mathrm{E_i}\big(-\mathrm{i}k(z + \mathrm{i}h)\big) - \frac{\mathrm{i}}{(z+\mathrm{i}h)^2} + C_1\Bigg)\mathrm{e}^{\mathrm{i}\omega t} + \Bigg(4k^2\mathrm{e}^{-\mathrm{i}k(z-\mathrm{i}h)}\mathrm{E_i}\big(-\mathrm{i}k(z - \mathrm{i}h)\big) + \frac{\mathrm{i}}{(z-\mathrm{i}h)^2} + C_2\Bigg)\mathrm{e}^{-\mathrm{i}\omega t}\Bigg] \tag{16}$$

Thus, the circulation-induced velocity at the locations of the vortices $z = -\mathrm{i}h$ is given by

$$U_v = \mathrm{i}\frac{\Gamma c}{2\pi}\Bigg[(4k^2\mathrm{E_i}(0) + C_1)\mathrm{e}^{\mathrm{i}\omega t} + \Bigg(4k^2\mathrm{e}^{-2kh}\mathrm{E_i}(-2kh) + \frac{1}{4h^2} + C_2\Bigg)\mathrm{e}^{-\mathrm{i}\omega t}\Bigg] \tag{17}$$

where the locally induced velocity singularity at the centre of the vortices has been removed. In addition, the non-local circulation-induced velocity tends to zero as the depth increases and so $C_1 = -4k^2\mathrm{E_i}(0)$ and $C_2 = 0$. Thus, the velocity is given by

$$U_v = \mathrm{i}\frac{\Gamma c}{2\pi}\left(4k^2\mathrm{e}^{-2kh}\mathrm{E_i}(-2kh) + \frac{1}{4h^2}\right)\mathrm{e}^{-\mathrm{i}\omega t} \tag{18}$$

Noting that the radial $u_{cr}$ and tangential velocities $u_{c\theta}$ in a coordinate system moving with the vortices are given by

$$(u_{cr} + \mathrm{i}u_{c\theta})\mathrm{e}^{-\mathrm{i}\omega t} = U_v \tag{19}$$

Thus, separating the velocity $U_v$ into radial and tangential velocities gives

$$u_{cr} = \frac{\Gamma c}{2\pi}\left(-4k^2\mathrm{e}^{-2kh}\Im_{\mathrm{m}}\big(\mathrm{E_i}(-2kh)\big)\right) \tag{20}$$

$$u_{c\theta} = \frac{\Gamma c}{2\pi}\left(-4k^2\mathrm{e}^{-2kh}\Re\big(\mathrm{E_i}(-2kh)\big) + \frac{1}{4h^2}\right) \tag{21}$$

One further step can be taken for the radial velocity as the Exponential Integral has the property that $\Im_{\mathrm{m}}\big(\mathrm{E_i}(x < 0)\big) = -\pi$ [18], so that the radial velocity is given by

$$u_{cr} = 2\Gamma ck^2e^{-2kh} \tag{22}$$

An approximation of the real part of the exponential integral [19] can be used to estimate the tangential velocity as

$$u_{c\theta} = \frac{\Gamma c}{2\pi}\left(-4k^2\mathrm{e}^{-2kh}\left(\gamma + \ln(2kh) + \sum_{n=1}^{\infty}\frac{(kh)^n}{n\,n!}\right) + \frac{1}{4h^2}\right) \tag{23}$$

Where γ is the Euler–Mascheroni constant (0.57721 to 5 decimal places). The forces due to these velocities are equal to the cross-product of the circulation and velocity and so the radial velocity will cause a tangential force and the tangential velocity will cause a radial force. Because the vortices move at a constant velocity about the axis of rotation, the tangential force is always in-phase with the vortex velocity and the radial force is always orthogonal to the vortex velocity, being either a centripetal or centrifugal force depending on the sign of the vorticity. Thus, the tangential force may be considered as the added damping force as it is always in phase with velocity and the radial force as the added mass force as it is always in phase with acceleration.

Thus, the tangential force (sum of both hydrofoils) due to motion of the two vortices is given by

$$F_{c\theta} = 2\rho\Gamma u_{cr} = 4\rho\Gamma^2ck^2\mathrm{e}^{-2kh} \tag{24}$$

This can be used to define an adding damping coefficient $B$ by dividing this force by the vortex velocity so that

$$B = \frac{F_{c\theta}}{\omega c} = 4\rho\frac{k^2}{\omega}\Gamma^2\mathrm{e}^{-2kh} \tag{25}$$

It is now possible to verify this derivation of the added damping coefficient by using the Haskind Relations [20], which links the wave excitation force from (3) to the added damping coefficient in (25). Haskind Relations are based on far-field considerations and so is not affected by the local hydrodynamics. That is, it is equally relevant whether considering a wave force / radiation due to a moving vortex as when considering a wave force / radiation due to a heaving buoy. Thus, using the Haskind Relation, noting that the force in (3) needs to be doubled to account for both vortices, the expression for the added damping should be

$$B = \frac{\omega}{\rho g^2\zeta^2}|F_i|^2 = \frac{\omega}{\rho g^2\zeta^2}2^2\rho^2\Gamma^2\omega^2\zeta^2\mathrm{e}^{-2kh} = 4\rho\frac{k^2}{\omega}\Gamma^2\mathrm{e}^{-2kh} \tag{26}$$

Which corresponds with (25), verifying the derivation of the potential flow solution and associated hydrodynamic forces.

## III. DEVELOPMENT OF FUNDAMENTAL UNDERSTANDING

The expressions developed for the excitation and radiation damping forces can be used to determine the maximum power capture and the conditions required for this to occur. To achieve this, it is convenient to consider the rotational mode around the axis of rotation. Unlike a 'conventional' wave energy converter, the tangential wave excitation force and the body velocity are both time invariant, which means that the power capture $P$ is constant and given by

$$P = (2\rho\Gamma c\omega\xi \mathrm{e}^{-kh}\sin\gamma - 4\rho\Gamma^2 c^2 k^2 \mathrm{e}^{-2kh})\omega \tag{27}$$

Two parameters that may be controlled to maximise the power capture are the circulation and the radius of rotation. However, it can be seen in (27) that it is possible to define a more generalised parameter, which is the product of the circulation and radius $\Gamma c$. Then, by differentiation of (27) with respect to the product of circulation and radius provides the following results for the optimum product of circulation and radius and the maximum power capture in this optimum condition

$$\frac{dP}{d(\Gamma c)} = (2\rho\omega\xi \mathrm{e}^{-kh}\sin\gamma - 8\rho k^2\Gamma c \mathrm{e}^{-2kh})\omega = 0 \tag{28}$$

So that

$$(\Gamma c)_{opt} = \frac{g^2}{4\omega^3}\sin\gamma\, \mathrm{e}^{kh}\xi \tag{29}$$

$$P_{max} = \frac{\rho g^2}{4\omega}\xi^2\sin^2\gamma \tag{30}$$

Equation (29) shows that there is an optimum product of circulation and radius that will maximise the power capture of a rotating lift-based wave energy converter. It can also be seen that the optimum product increases with the wave height and depth of submergence, whilst decreasing significantly (a cubic relationship) with the wave frequency. In many designs it is likely that the radius of rotation is fixed and so it is necessary to use the optimum circulation to maximum the power capture. However, it is important to note that the maximum power is not generated by maximising the circulation or wave excitation force. This is because, an overly large circulation will increase the amount of radiated wave energy faster than the increase in absorbed energy leading to a reduction in the power capture.

Finally, it is possible to calculate the proportion of the incident wave energy that can be extracted by the system as the incident wave power density is equal to $\rho g^2\xi^2/4\omega$. Comparing this to (29) shows that the maximum proportional of wave energy capture occurs when the phase angle between the wave and the location of the vortices is ±90° ($\sin^2\gamma = 1$) and in this case 100% of the wave energy can be extracted from the incident waves. This is possible because although there is only a single independent degree of freedom, the vortices are effectively moving in heave and surge as it rotates about its axis. Moreover, the amplitude and phase requirements for the motions in heave and surge are satisfied exactly in the case of deep water by a constant rotation about a fixed axis. Although, it has been identified that this ideal circular motion of the vortices is not generally true for neither finite water depths nor finite hydrofoil lengths [21], this is the case in these idealised circumstances and provide the necessary simplification of the hydrodynamics to support design. A similar result showing a 100% extraction of wave energy has been previously shown for a 'conventional' wave energy converter moving in a circle [22] as well as for a lift-based wave energy converter using potential flow models that include harmonics of the wave frequency due to a finite amplitude of vortex motions [9], [23].

These findings can be usefully compared to the fundamental requirements to maximise the power capture of a 'conventional' wave energy converter. A traditional wave energy converter has the joint conditions of the optimum phase and amplitude of motion to maximise power capture [24]. These same conditions can be seen in the optimum control of a rotating lift-based wave energy converter, albeit in a slightly different form. Specifically, the optimum phase condition can be seen as the requirement to have a 90° phase difference between the rotation of the vortices and the incident wave. This phase relationship ensures that the force due to the incident wave acts purely tangentially and so maximises its effectiveness in driving the rotation of the vortices. The optimum amplitude of motion condition has been replaced by an optimum product of circulation and radius condition. This provides more flexibility in the design of a rotating lift-based wave energy converter as either the circulation or radius can be adjusted to achieve the optimum condition. Of course, whether this additional flexibility can be readily exploited is a more complex proposition, but this is considered to be outside of the scope of this paper.

An encouraging finding is that unlike 'conventional' wave energy converters, a rotating lift-based wave energy converter does not depend on having a natural frequency equal to the wave frequency to maximise power capture. There is an optimum phase requirement for maximum power capture, but this can be satisfied without the need to balance the reactive energies in the system. This is because in a rotating system with a fixed angular velocity the reactive energy is constant and so tuning using an appropriate natural frequency is not required to achieve the optimum phase of motion relative to the wave. The vortices do have an 'added mass' associated with their motion, which manifests as a purely radial force as this is the direction of acceleration in the global (inertial) reference frame. It is worth noting that in the absence of a free-surface, the lift force generated by the motion of the vortices is purely radial as all the velocities are purely tangential. If this lift force is divided by the radial acceleration required to keep the vortices moving in a circle it can be seen as the equivalent to the infinite-frequency added mass term. The tangential velocities induced on the vortices by the presence of the water surface are given by Equation (21) are functions of the wave number and so are associated with the frequency dependence of the added mass, as typically seen for floating bodies. The variation in the radial velocities induced by the vortices is responsible for the frequency dependence of the added damping coefficient.

A particularly powerful finding that arises from this analysis is that it is the incident velocities, due to the incident wave, the vortices' motion, and the generated waves, that define the forces on the body. In retrospect, this may be considered to be a relatively obvious finding for this simplified model (although perhaps not so obvious prior to the analysis), but it is considered important to state. Especially because identifying this relationship provides additional insight into the generation of circulation, which is discussed in the following section.

## IV. GENERATION OF CIRCULATION

The analyses described in the sections above assume that there is a circulation but include no consideration of how this circulation is generated. Circulation can be generated about a body either actively, for example by spinning the body, which generates circulation dependent on the speed of the spin as for example in a Magnus Rotor [25], [26], or passively, for example by use of a hydrofoil that generates circulation dependent on the incident velocity and angle of attack [27]. In the case of active generation of circulation, the magnitude of circulation is likely to be relatively insensitive to the surrounding fluid conditions and so the above analysis can be interpreted without too much need for refinement. However, in the case of passive generation of circulation, the magnitude of circulation is likely be influenced by the change in surrounding fluid conditions that have been generated by the circulation.

Where hydrofoils are used to generate the circulation, provided the hydrofoils are outside the region of stall, then the amount of circulation is approximately proportional to the product of the incident velocity and the angle of attack [27]. In this case, the incident velocity is the vector sum of velocities of; the water particle velocity due to the incident wave, the hydrofoil velocity due to its rotation around the axis and the water particle velocity induced by the circulation. In general, the velocity due to the rotation of the hydrofoil around its axis is around an order of magnitude larger than the other velocities and so this can often be used as the incident velocity for a conceptual design study. However, at the optimum phase relationship the incident wave-induced water particle velocity and the radial vortex-induced water particle velocity are both orthogonal to the velocity due to the hydrofoil's motion making the hydrofoil angle of attack more sensitive to these other velocities. In particular, the radial vortex-induced water particle velocity opposes the incident wave-induced water particle velocity so that the angle of attack will be smaller than that if the effect of this velocity were ignore. Moreover, we know from the maximum power transfer theorem that at maximum power capture the radiation damping force should be equal to half the incident wave force, which can be shown by simply substituting Equation (29) into Equations (24) and (3) for the radiation damping force and incident wave force respectively. Consequently, the vortex-induced radial velocity would be equal to half of the wave induced velocity, which would result in a halving of the angle of attack compared to the case where it is not considered, with an associated reduction in circulation. Thus, the insight that the radiated wave will induce a velocity that will reduce the angle of attack of the hydrofoil provides clear guidance on how the pitch of the hydrofoil may need to be set and controlled to maximise power capture.

## V. CONCLUSIONS

It is shown that it is possible to generate a linear potential flow solution for a two-dimensional rotating lift-based wave energy converter by using the relatively standard assumptions that the wave amplitude and body motions are infinitesimally small. This linearised solution had not previously been produced. It is found that linear potential flow solution results in conditions for maximum power capture that have the same fundamental structure as for a wave energy converter excited by Froude-Krylov and diffraction forces, except that the amplitude of body motion is replaced by the product of the circulation and radius of rotation. It is also shown that at the optimum product of circulation and radius of rotation, 100% of the incident wave energy can be extracted from the waves.

The similarity of the rotating lift-based wave energy converter linear potential flow solution to that for a wave energy converter excited by Froude-Krylov and diffraction forces means that many of the well-known findings that have traditionally been used to analysis and design wave energy converters are also relevant to the design of rotating lift-based wave energy converters. For example, it was shown [6] that in the case that the optimum amplitude of motion could be not achieved and is $R$ of the optimum amplitude of motion, then the maximum percentage of energy that can be extracted from the wave $\eta$ is given by

$$\eta = 2R - R^2 \tag{31}$$

It can easily be shown that the same expression can be written for a rotating lift-based wave energy converter, except that $R$ refers the proportion of the optimum product of circulation and radius that can be achieved. Thus, rotating lift-based wave energy converters have the same relative insensitivity to the optimum conditions as 'conventional' wave energy converters. For example, if only 70% of the optimum product of circulation and radius can be achieved then the maximum potential power capture reduces by only 9%. Moreover, this insensitivity is symmetric in that if the actual product of circulation and rotation is greater than the optimum then there is a similarly gradual reduction in the power capture (a finding that was not highlighted by Evans due to how his equations were constructed). This is particularly important for rotating lift-based wave energy converters as it is possible that too much circulation will be generated during some periods when operating in real seas and this result indicates that this should not cause a dramatic reduction in the power capture.

Finally, although a linear hydrodynamic model of a wave energy converter provides valuable insight into its performance that can help with its conceptual design, it is also important to recognise that there may be circumstances where the combination of hydrodynamic characteristics means that the insight needs to be treated with caution. This was shown to be the case for oscillating wave surge converters (OWSCs), where the maximum capture width ratio for an OWSC may not necessarily occur when the WEC has been designed to be resonant [28]. It is not that insights from a linear hydrodynamic model are wrong, but that they may need refinement to account for real circumstances that include a range of non-linearities. For rotating lift-based wave energy converters it is anticipated that further refinement of the model will be required to account for expected non-linearities.

A non-linearity that needs to be carefully considered in the design of a many rotating lift-based wave energy converters is the generation of circulation. It has been shown that the circulation itself will influence the surrounding fluid conditions, which may have an impact on the generation of the circulation in the case of passive generation of circulation by, for example, a hydrofoil. Another non-linearities that is expected to be important in this respect is the generation of drag, which is likely to change the optimum conditions for maximum power capture and result in a reduction in power capture. This will be the subject of a following paper.

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