Misled by Betz and unsteady flow – review on turbine arrays falsely deemed 'optimal'

Peter F. Pelz, Jan Lemmer, Christian B. Schmitz

***Abstract*—A turbine array is an adjustable flow resistance R placed in a tidal channel. Ideally, it is designed and operated to maximise energy yield. Garrett & Cummins (2005), using optimal control theory applied to the RC-element channel (R) and basin (C), showed: the energy extraction from the flow $P_\mathrm{T} + P_D$ is maximised when the flow rate is slowed down by a factor of $1/\sqrt{3}$. This result is independent of the ratio of the extracted mechanical power P_T to the total power extraction including the power loss P_D due to the mixing of the bypass flows within the turbine field. The optimisation task for turbine arrays is maximising P_T. This objective raises two questions: "What is the maximum power P_T that can be extracted, and what is the optimal design (size, topology) and operation to achieve this output?" When addressing them, the literature still uses the Betz 'limit' as a reference. The work presented highlights two major problems. First, the Betz 'limit' is not a constant upper bound for open channel flow. This problem has been discussed and solved by the first author (2011, 2020). Second and more importantly, the presented paper points out the misconception under which several research studies referred to array topologies as 'optimal' with regard to design and operation. Hereby, the presented paper contributes to the advancement of tidal power on an axiomatic basis. The misleading by Betz and overvaluing of transient effects is made transparent in a scientific discourse.**

***Index Terms*—Betz limit, blockage, coefficient of performance, optimisation, quasi-steady flow, tidal array, tidal fence, turbine array**

I. Introduction

THE aim of modelling tidal turbines is to provide designers and engineers with guidelines where to build, how to arrange and how to operate tidal turbines in a given tidal channel. Tidal turbines as a component of a technical system are embedded in a complex socio-technical environment, cf. Fig. 1. Each system defined by a system boundary is characterised by objectives and constraints. The inner boundary encloses the material choice. The component system is focused on the individual component, e.g. one individual turbine without considering the bypass or the added resistance to the channel. In the techno-economic ecological system, the quality measures costs to society as well as value to society are either objectives or constraints [1], [2]. The outermost system boundary encloses the socio-technical system including the different stakeholders.

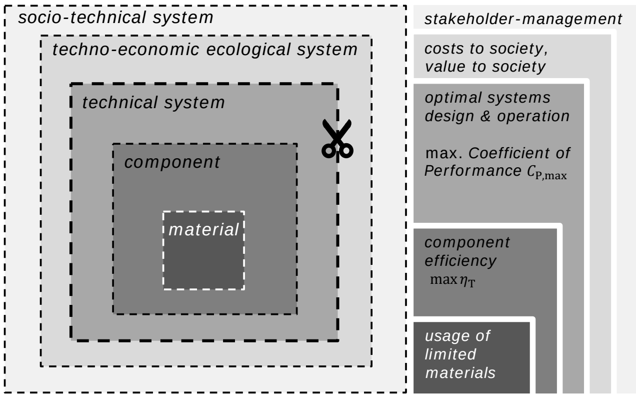

Fig. 1. Technical systems are embedded in a socio-technical environment of stakeholders. The focus of this paper is the optimal design & operation of the technical system consisting out of the tidal channel, the basins and the turbine array.

In this paper we focus the discussion on the technical system consisting out of the tidal channel, the basins and the turbine array. For this technical system maximising the harvested energy W_T over one tidal cycle with the cycle time T is the major point of interest and results in the optimisation problem

$$\max W_\mathrm{T} = \max \int_0^T P_\mathrm{T} \, \mathrm{d}t. \tag{1}$$

With P_T as the array's power output which is the sum of the individual turbine power outputs.

In this paper it is clarified why using the Betz coefficient of performance $C_{\mathrm{P,Betz}}$ can be misleading regarding the optimal operation and design of an array. Governing physical principles in a tidal channel are highlighted. These need to be taken into account when optimising the design and operation of an array. In order to answer this, we address four detailed questions:

1) Can the flow in a tidal channel be modelled as quasi-steady?

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2) How can a two-stage optimisation problem for design and operation be formulated?
3) Are the energy references chosen correctly to solve the optimisation problem?
4) Are the used actuator disc models physically consistent and hence suitable to answer questions 1 to 3?

Question 1 will be investigated by an order of magnitude analysis based on dimensional analysis in section II of the paper. The second question will be answered by a formal derivation of a two-stage optimisation problem based on (1). It is clarified in section III why using the Betz coefficient of performance $C_{\mathrm{P,Betz}}$ can be misleading regarding the optimal operation and design of an array and an answer to question 3 is given. In section IV, question 4 is answered by recapturing recent findings about the physical consistency of current turbine and array models. Finally, an assessment of current approaches from an optimisation view is done.

II. Condition for quasi-steady flow

In order to answer question 1, it is essential to understand the conditions for quasi-steady flow. In steady flow, the flow values are only a function of the position. Whereas in transient flow they are a function of the position and time. If the temporal change follows a pattern the complexity can be reduced (e.g. harmonic flow).

The time dependencies in dynamics are summarized in Fig. 2. If the changes are so slow that the system reacts immediately to the changes in boundary conditions it is called quasi-steady [3]. Typically for quasi-steady problems the time t does not enter the model equation explicitly in form of a differential operator but only implicitly as a time dependent parameter in the boundary conditions.

There are two necessary conditions to be met for a system to be named quasi-steady. Firstly, the excitation frequency must be much smaller than the system's lowest natural frequency: $f \ll 1/\sqrt{IC}$. Here I is the inductance representing liquid body and C is the capacity or compliance associated with the storage of mass and/or energy. The latter is here, due to the evaluation of the free surface.

The flow through a tidal channel is a function of time: in the northern hemisphere tides are usually semidiurnal, this translates into a cycle time of $T = f^{-1} \approx 4.5 \times 10^4\,\mathrm{s}$. For a channel of depth $h_0 \sim 10^1\,\mathrm{m}$ and length $L \sim 10^4\,\mathrm{m}$, the order of magnitude of the natural frequency is $\sqrt{gh_0}/L \sim 10^{-3}\,\mathrm{s}^{-1}$, which is two orders of magnitude larger than $f \sim 10^{-5}\,\mathrm{s}^{-1}$.

Secondly, for a diffusion process, the excitation frequency f must be much smaller than the diffusion time of the process for a quasi-steady flow. Here, it is the diffusion of turbulent shear stress τ_b across any cross section of the tidal channel or along a mixing layer due to a change in the boundary condition. The order of magnitude of this diffusive relaxation process

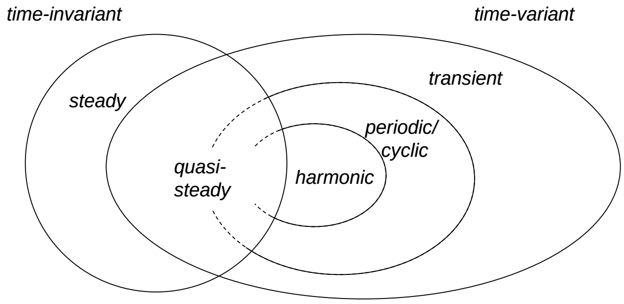


Fig. 2. Euler diagram for classification of time-variant and time-invariant processes, adopted from [2].

is determined to be ν_t/h_0^2 with the eddy viscosity ν_t and the water depth h_0. The order of magnitude of the eddy viscosity ν_t is given by the product of frictional velocity $u_* = \sqrt{\tau_b/\varrho}$ and water depth, $\nu_t = \kappa\, u_* y \sim u_* h_0$. Here, $\kappa = 0.4$ is the von Kármán constant. From von Kármán's log-law

$$\frac{u_0}{u_*} \approx \frac{1}{\kappa} \ln \frac{h_0}{k} + 5, \tag{2}$$

we derive the order of magnitude estimation for the frictional velocity $u_* \sim (1 \dots 100)\,\mathrm{cm/s}$ for a seabed roughness of $k \sim (1 \dots 10)\,\mathrm{cm}$ and a undisturbed flow velocity $u_0 \sim (1 \dots 10)\,\mathrm{m/s}$.

To summarise, the two necessary conditions to be met for a quasi-steady channel flow are written as two dimensionless products

$$\varepsilon = \frac{f}{\sqrt{IC}} = \frac{fL}{\sqrt{gh_0}} \sim 10^{-2}, \tag{3}$$

$$Wo^2 = \frac{fh_0^2}{\nu_t} \sim \frac{fh_0}{u_*} \sim 10^{-2}. \tag{4}$$

The shallow water equations written in dimensionless form, shown in Sec. V, can also be posed as regular perturbation problem, with the perturbation parameter ε and the stationary problem as an undisturbed problem. The parameter Wo is known as the Womersley number in biofluid mechanics [4]. It describes the evolution of the velocity profile in an oscillating flow. It should be noted that the evolution of velocity profiles is not modelled by the shallow water equations (SWE). It only appears in a 3D flow model. Hence, by solving the SWE the diffusion effects are ignored right from the very beginning.

Both perturbation parameters are much smaller than 1. Hence, the flow is indeed quasi-steady. Here, it has to be stressed that the transient effects due to the inductance of the flow are of minor importance and usually negligible due to the smallness of the perturbation parameter $\varepsilon \sim 10^{-2}$. Only in the special case of a very long and at the same time shallow channel, does the wave cycle time share the same order of magnitude with the tidal cycle time T. It must therefore be treated as transient [5]. This is consistent with the findings of Garret & Cummins [6], Pelz et al. [5] and contradicts the recent paper from

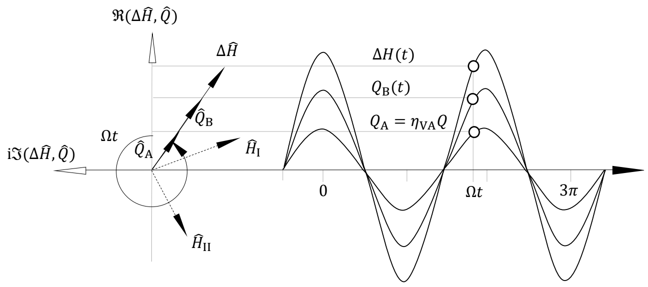


Fig. 3. Electrical analogy to a quasi-steady tidal channel flow for $\varepsilon \ll 1$, $Wo \ll 1$.

Bonar et al. [7].

The electrical analogy is most suitable for demonstrating the findings illustrated in the above even though this analogy is valid only for linear systems. Here, we fully recognise that the flow in a tidal channel and the flow through a turbine array is nonlinear and therefore a linear model is not an appropriate model. This is due to the nonlinearity of the convective term in any transport equation. In other words, the dissipative character of the flow is nonlinear, i.e. the resistance elements of the tidal channel. Nevertheless, the role of inductance and compliance can well be understood in the electrical analogy. In a linear system the difference of the total head $\Delta H = H_\mathrm{I} - H_\mathrm{II}$ between the two infinitely large basins drives the flow, cf. Fig. 4. Again, this is true only for linear systems. Researchers and engineers would be misguided in considering this difference to be the driving force when attempting to capture the important nonlinear effects of the hyperbolic system. In fact, as in gas dynamics, the head ratio $H_\mathrm{II}/H_\mathrm{I} = h_\mathrm{II}/H_\mathrm{I}$ is physically more appropriate. This head ratio gives the downstream boundary condition to the flow [5].

The first harmonics with $\Omega = 2\pi f$ of the head difference $\Delta H \approx \Re(\hat{H}\mathrm{e}^{\mathrm{i}\Omega t})$ and the two cyclic volume flows, Q_A through the array and the bypass volume flow Q_B, are given by the Fourier transform

$$\Delta\hat{H} = \frac{1}{\pi}\int_0^{2\pi} \Delta H\,(\Omega t)\,\mathrm{e}^{-\mathrm{i}\Omega t}\;\mathrm{d}\Omega t, \qquad \hat{Q}_\mathrm{A,B} = \frac{1}{\pi}\int_0^{2\pi} Q_\mathrm{A,B}\,(\Omega t)\mathrm{e}^{-\mathrm{i}\Omega t}\;\mathrm{d}\Omega t. \tag{5}$$

For $\varepsilon \ll 0$ the volume flow is in phase with the head $\Delta\hat{H}$ as depicted in Fig. 3, i.e.

$$\arg\hat{Q}_\mathrm{A}/\Delta\hat{H} = \arg\hat{Q}_\mathrm{B}/\Delta\hat{H} = 0. \tag{6}$$

III. Array Optimisation

We now address the second question of this paper, the proper formulation of the optimisation problem. As it is common in engineering science, the optimisation problem has an objective and several constraints. The constraints are given by physical laws such as the energy equation and continuity equation as well as the boundary conditions. Further constraints are functional, ecological and logistic requirements and available technologies.

The design task based on eq. (1) is written as

$$\max W_\mathrm{T} = \max \int_0^T P_\mathrm{T}(d, o(t), c(t))\;\mathrm{d}t, \quad \text{s.t. physical laws.} \tag{7}$$

The turbine power $P_\mathrm{T}(t)$ is a function of the array design d, the operation $o(t) = o(t+T)$ such as the turbine volume flow $Q_\mathrm{T}(t)$ or the turbine head $H_\mathrm{T}(t) := P_\mathrm{T}/(\varrho g Q_\mathrm{T}(H_\mathrm{I} - H_\mathrm{II}))$. The transient boundary conditions $c(t)$ are given by the energy height $H_\mathrm{I}(t)$ or $H_\mathrm{II}(t)$. To give an example, Pelz et al. [5] use for a tidal turbine with bypass the blockage σ as design, the turbine head H_T as operation and either the downstream Froude number Fr_2 or the dimensionless water depth $\bar{h}_2 := h_2/H_\mathrm{I}$ as boundary conditions.

The boundary conditions are given by nature and can not be influenced by the engineer. Thus for practical optimisation, the design d and operation $o(t)$ over the tidal cycle need to be in focus. This may be formulated as a two-stage constrained optimization problem.

It is two-stage since first the design has to be optimised anticipating the operation $o(t)$ and the boundary conditions $c(t)$. This is typical for many engineering problems [2], [8]. While the operation parameter can be adapted after the installation, the design is fixed with construction. Therefore an optimal design for the whole cycle needs to be found. Here the constrained optimisation problem (7) is in fact an optimal control problem for the operation as a function of design and boundary conditions. The optimal control problem is equivalent to the variational problem [9]

$$\delta\int_0^T P_\mathrm{T}(d, o(t), c(t))\mathrm{d}t. \tag{8}$$

The solution of this variational problem (8) is equivalent to a solution of the Euler-Lagrange equation [10, p. 184f.]

$$\frac{\partial P_\mathrm{T}(d, o(t), c(t))}{\partial o} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial P_\mathrm{T}(d, o(t), c(t))}{\partial \dot{o}}\right). \tag{9}$$

The 'right hand side' of the Euler-Lagrange equation (9) is zero, as long as $Wo \ll 0$ and $\varepsilon \ll 0$ (cf. Sec. II). Hence the optimisation problem reduces to

$$\frac{\partial P_\mathrm{T}(d, o(t), c(t))}{\partial o} = 0. \tag{10}$$

In a fully obstructed channel the two-stage problem reduces to a single-stage (operation) problem, which can be solved analytically [6], [9]. For turbines with bypass flow (10) needs to be solved numerically or by using an analytical approximation for P_T.

The operation parameter $o(t)$ for the general array problem is described by all individual turbine heads

$H_{\mathrm{T}l,k}(t)$. $H_{\mathrm{T}l,k}(t)$ is the turbine head of the (l,k)-th turbine in a turbine array with L rows and K turbines per row. An alternative way to describe the operation is to use the volume flow rate through the channel $Q(t)$, as used by Schmitz & Pelz [9]. The advantage of this is, that the volume flow is the same for all turbines in one column whereas the optimal turbine head differs for each turbine. Assuming that the turbines reduce the volumetric flow in an optimal way, i.e. $H_{\mathrm{T}l} = H_{\mathrm{T}l,\mathrm{opt}}$ [11], the number of operational parameters is drastically reduced from L to 1. As a result of the optimisation the optimal design d_{opt} and the optimal operation $o_{\mathrm{opt}}(t)$ are calculated.

IV. Reference Energy

In the following section, we answer the question: 'Are the energy references chosen correctly to solve the optimisation problem?'. We show that $C_{\mathrm{P,Betz}}$ does not use a reasonable reference energy scale in the case of tidal power. First, a dimensional analysis is presented and then the question is answered whether the energy references are chosen correctly for solving the optimisation problem (7).

The performance rating of a tidal turbine or a tidal fence is based on the coefficient of performance C_{P} which is given by the ratio of extracted mechanical power P_{T} to the available power P_{avail}. The coefficient of performance is also used as objective for the optimisation in many studies. It was introduced by A. Betz over 100 years ago for wind turbines [12], while F. W. Lanchester used a similar approach for analysing propellers before [13]. The reference energy scale of a wind turbine is the kinetic energy of the undisturbed free stream flow, thus the available power is given by

$$P_{\mathrm{avail,Betz}} := \frac{\varrho}{2} u_0^3 A_{\mathrm{T}}. \tag{11}$$

With the turbine area A_{T}, the free-stream velocity u_0 and the fluid density ϱ. Betz provides the upper limit for the power extraction of an ideal wind turbine as $(C_{\mathrm{P,Betz}})_{\max} = 16/27$.

A. Dimensional analysis

Figure 4(a) shows the top view of an array in a tidal channel schematically, whereas Fig. 4(b) shows the corresponding cross-sectional view. The overall power output of all turbines $P_{\mathrm{T}}(t)$ of the array being described by a lateral array blockage, $0 \le \sigma_{\mathrm{A}} \le 1$, and length factor $0 \le \lambda \le 1$. Depending on the structure of the array and the number of rows, further design parameters such as the local blockage $0 \le \sigma_l \le 1$ are required. All these parameters are summarized in d for simplicity. The downstream boundary condition is given by the water depth h_{II} at the outlet of the tidal channel. For subcritical flow, this depth equals H_{II} due to Newton's second law 'actio est reactio' [5]. The power output and therefore the coefficient of performance depends further on the surface averaged friction factor c_f multiplied with the seabed area LB and the turbine operation parameter given by the

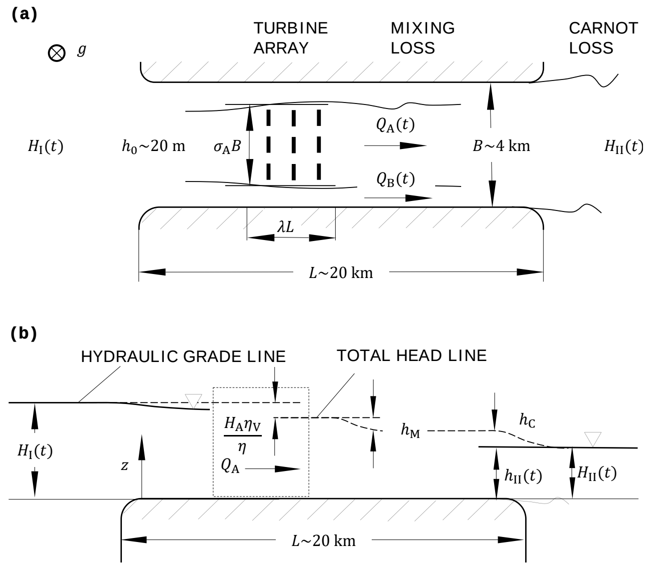


Fig. 4. Schematic of a turbine array covering the width $\sigma_{\mathrm{A}} B$ and length λL of the tidal channel; downstream of the turbine array the mixing of array volume flow $Q_{\mathrm{A}}(t)$ and bypass volume flow $Q_{\mathrm{B}}(t) = Q(t) - Q_{\mathrm{A}}(t)$ takes place; (a) top view (b) side view of the channel.

volume flow $Q_{\mathrm{A}}(t)$ through the turbine array or the turbine head $H_{\mathrm{A}}(t)$ representing the power extraction of the whole array.

Inertia is negligible for a typical tidal channel, because the conditions for quasi-stationary flow apply, as shown in Sec. II. Hence, time only appears parametrically in the boundary condition $h(t)$, $H_{\mathrm{I}}(t)$ but not explicitly.

For reasons of dimensions [14], [15] the coefficient of performance in the quasi-steady flow is given as a result of a dimensional analysis

$$C_{\mathrm{P}} = \mathrm{fn}\left(\underbrace{\sigma_{\mathrm{A}}, \sigma_l, \lambda, \ldots}_{d}, \underbrace{\overline{H}_{\mathrm{A}}}_{o(t)}, \underbrace{\overline{h}_{\mathrm{II}}}_{c(t)}, \frac{L}{H_{\mathrm{I}}} c_f\right), \tag{12}$$

with the dimensionless measures

$$C_{\mathrm{P}} = \frac{P_{\mathrm{T}}}{P_{\mathrm{avail}}} = \frac{P_{\mathrm{T}}}{2\varrho g^{1/2}(2/5 H_{\mathrm{I}})^{5/2} B} \le \frac{\eta_{\mathrm{A}}}{2}, \quad \overline{H}_{\mathrm{A}} = \frac{H_{\mathrm{A}}}{H_{\mathrm{I}}},\ \overline{h}_{\mathrm{II}} := \frac{h_{\mathrm{II}}}{H_{\mathrm{I}}}, \tag{13}$$

and the available power [16]

$$P_{\mathrm{avail}} := 2\varrho B g^{3/2} \left(\frac{2}{5} H_{\mathrm{eff}}\right)^{5/2}. \tag{14}$$

The coefficient of performance is limited to $1/2$ by any means as shown by the first author of this paper [16].

Now the question remains if the constitutive equation for c_f does depend on time. This is not the case, as long as the diffusion time $H_{\mathrm{I}}^2/\nu_t \sim h_0^2/\nu_t$ is much shorter than the cycle time $T = 12.42\,\mathrm{h}$. With this the diffusion time is of the order $h_0^2/\nu_t \sim h_0/u_* \sim 10^3\,\mathrm{s}$. This is in any case much shorter than the cycle time:

$$\frac{f h_0^2}{\nu_t} \ll 1. \tag{15}$$

B. Reference energy scale in tidal channels

A reasonable reference energy scale of a tidal turbine is not the same as for wind turbines due to the free surface flow. This influence is only negligible if the turbine blockage σ is small and the Froude number Fr tends toward zero. Then a hydrokinetic turbine in free surface flow can be treated with Betz's disk theory and thus the velocity u_0 is an appropriate basis for an energy scale. Otherwise, the effective height $H_{\text{eff}} := H_0 + z_{\text{I}} - z_{\text{II}}$ with the specific energy height $H_0 = h_0 + u_0^2/2g$ above the base height $z = z_{\text{I}}$ before the array is a suitable and functional choice for an energy scale [16]. The base height after the array z_{II} is usually equal to $z_{\text{I}} = z_{\text{II}} = z$ and is therefore omitted. The energy height forms the counterpart to the boiler state in gas dynamics, i.e. it is $H_0 = H_{\text{I}} = h_{\text{I}} + u_{\text{I}}^2/2g$.

The first law of thermodynamics shows that for a turbine efficiency of one and a fully obstructed channel, the maximum mechanical power that can be extracted from the flow is [16]

$$P_{\text{T,max}} = \varrho B g^{3/2} \left(\frac{2}{5} H_{\text{eff}}\right)^{5/2}. \quad (16)$$

This requires an optimal control of the turbine head H_{T} for a given H_{I}, with the volume flow Q_{T} through the turbine. The operation is optimal for $\sigma = 1$ provided that $\overline{H}_{\text{T}} := H_{\text{T}}/(\eta_{\text{T}} H_{\text{eff}}) = 2/5$ is set. At this operating point, the unusable energy flux due to underwater flow and the utilized exergy flux are equal and the total, hypothetical usable, exergy current provided by nature is given by Eq. (14).

As in Betz [12], the available power is defined by a hypothetical machine with no downstream, i.e. an ideal energy sink. The coefficient of performance is therefore at most half of the turbine efficiency $C_{\text{P}} := P_{\text{T}}/P_{\text{avail}} \leq \eta_{\text{T}}/2$ [16]. This asymptote is independent of the design d and operation $o(t)$ of the turbine field.

A comparison between the two definitions of the coefficient of performance is shown in Fig. 5. The Betz coefficient of performance provides no constant upper limit and it can be shown that $C_{\text{P,Betz}}$ is even singular for $\sigma = 1, Fr_0 \to 0$ [5]. To act as a reference value, C_{P} needs to be independent from the Froude number Fr_0. Otherwise values for $C_{\text{P,Betz}}$ greater than the Betz-limit of $(C_{\text{P,Betz}})_{\max} = 16/27$ and also greater like one arise. This is commonly known and has been described by many authors [17]–[20]. Nevertheless the Betz coefficient of performance is still widely used in the tidal community, but gives no orientation as an upper limit should do.

It is obvious that definition (14) provides a reasonable reference whereas the Betz coefficient of performance is only useful for small blockages $\sigma_l \ll 1, \sigma_A \ll 1$ in channels with a low Froude number.

The second disadvantage of the $C_{\text{P,Betz}}$ becomes apparent when it is used for the optimization of tidal arrays that have high blockage ratios. In this case, the volume flow through the channel depends on the operation of the array and the free surface influence is not negligible. Thus the assumption of an unaltered flow is invalid for such an array and the velocity u_1 far in front of the array, that is often used for calculating $C_{\text{P,Betz}}$, changes depending on the array operation. Therefore optimal coefficient of performance does not necessarily lead to optimal operation in terms of maximized power output. This was shown by [21], [22] and is highly misleading for developers.

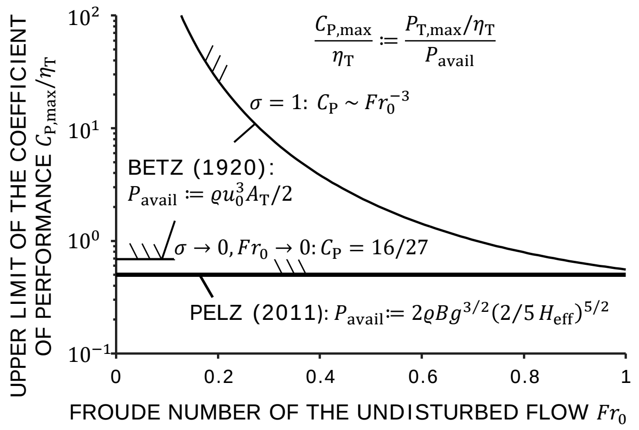


Fig. 5. Upper limit of the coefficient of performance versus Froude number of the undisturbed flow $Fr_0 = u_0/\sqrt{gh_0}$. The thick line shows C_{P} as defined by Pelz (2011). The coefficient of performance is independent of the Froude number and has a constant upper limit, whereas there is no Betz-limit for $\sigma > 0$ (adapted from [5]).

V. Assessment of current approaches

In order to answer the question 'are the used actuator disc models physically consistent and hence suitable to answer questions 1 to 3' we analyse five selected array modelling approaches. A short recapture of current models for both turbine and arrays is given before we discuss their optimisation objective, boundary conditions and results.

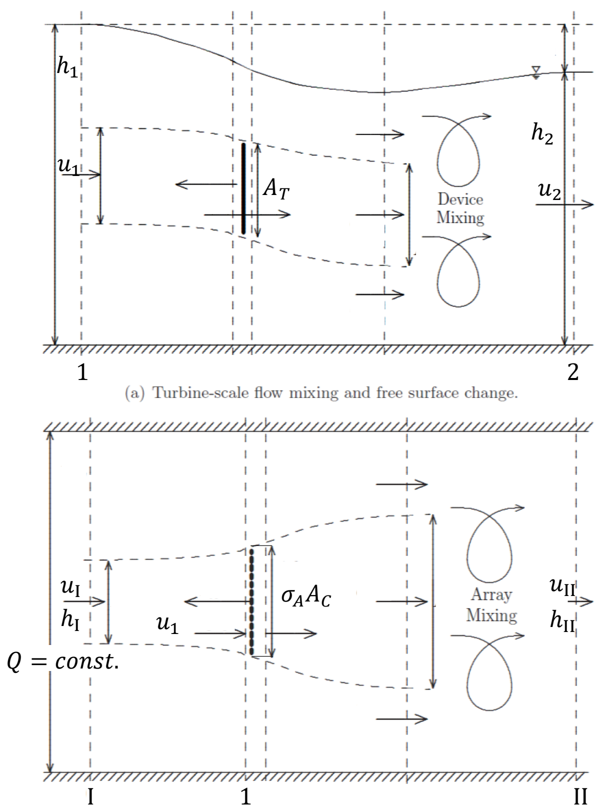
A. Recapture of modelling approaches for tidal arrays

There are tree models for tidal turbines: First (i) the model of Garret & Cummins [23]. Secondly the model introduced by Houlsby et al. [24] and Whelan et al. [17] which we refer to as model (ii). The third model (iii) was introduced by Pelz et al. [5].

The Models (i)-(iii) for a single turbine are also capable of describing an array that spans the whole channel width as shown by the above mentioned authors. An extension to model (i) is the work of Vennell [25] for an array spanning the whole channel width, which is a combination of model (i) and the channel model of Garret & Cummins [6].

So far, arrays completely blocking the whole channel width ($\sigma_A = 1$) were considered. The optimisation problem (10) for a partial array ($\sigma_A < 1$) depends not only on the operation and the array design in terms of array blockage σ_A but also on the array arrangement, especially the intra-turbine spacing, which can be described with the local blockage σ_l [18]. For a given array size ($\sigma := \sigma_A \sigma_l$) and a known boundary condition $c(t)$, e.g. h_{II} or Fr_{II}, both arrangement and operation need to be optimised. Nishino & Willden's model for partial arrays is based on the turbine model (i). It was extended in a following paper [19] to include the expansion of the turbine stream-tubes due to the array bypass flow.

The first partial array model that includes free surface influences was proposed by Vogel et al. [20] and is based on turbine model (ii), cf. Fig 6. This model does not consider the stream-tube expansion as introduced by Nishino & Willden [19].



(a) Turbine-scale flow mixing and free surface change.

(b) Array-scale flow mixing.

Fig. 6. Top view (b) of the array from Vogel et al. (model III) and side view (b) of the associated turbine model (ii). The authors assume the incoming volume flow as unaltered by the array operation, i.e. Q is constant. (adopted from [20])

The first model that combines free surface effects, stream-tube expansion as well as channel dynamics is from Gupta & Young [26]. It is an extension and combination of the models by Garret & Cummins [6], Nishino & Willden [18], [19] and Vogel et al. [20]. Bonar et al. [7] recently presented a numerical approach which combines the solution of the shallow water equations for the large scale calculation with the actuator disc model (ii) on the turbine scale. They consider the free surface influence as well as the channel dynamics and partly blocking arrays in one approach. They found a beneficial effect of the oscillating flow on the power generation for a specific tidal channel.

B. Discussion

Current studies on tidal arrays differ in objective and physical constraints. The latter is subdivided into model approaches and boundary conditions. Many studies provide an isolated examination of a physical effect, for instance free surface effects [17], local blockage [18], [19] and channel dynamics in tidal arrays [25]. The studies usually present an 'optimal' array. Yet each of the proposed array designs or operations are only optimal if a special case applies and not an optimal solution of Eq. (7), due to the following reasons:

- Neglecting guiding physical effects in the array and the surrounding channel.
- Ignoring the disturbance of the flow due to the array design and operation.
- Misinterpreting transient effects.

An overview of the selected studies is given in Table I, we will discuss them from an optimisation point of view. For clarity, upper Roman numerals are used to reference the array models.

The objective for optimising a tidal array is given by Eq. (7). For studies that assume steady flow (I) & (III), the integration over the tidal cycle vanishes and the objective is simplified to $\max P_{\mathrm{T}}$. This is further reduced to

$$\max C_{\mathrm{P,Betz}}(d, o(t), c(t)). \tag{17}$$

This is only valid if P_{avail} is not a function of $d, o(t), c(t)$, which is only the case for unaltered flow as assumed by model (I) & (III). Otherwise, even in steady flow, the reference velocity u_1 that is used to calculate $C_{\mathrm{P,Betz}}$ is influenced by the array operation and design, as shown in section IV. Thus maximising $C_{\mathrm{P,Betz}}$ does not necessarily led to $P_{\mathrm{T,max}}$, when the assumption of unaltered flow is changed to head-driven flow for a further investigation, as shown in [21], [22]. This was avoided in model (IV) and (V) by using the unaltered (flow without the array) flow velocity u_0 instead of u_1 as energy reference to calculate the Betz coefficient of performance and P_{avail}. Vennell uses a slightly different objective by optimising the product $(C_{\mathrm{P,Betz}} u_{\mathrm{max}}^3) \propto P_{\mathrm{T}}$, to avoid the before mentioned influence of design and operation on $C_{\mathrm{P,Betz}}$ [22]. All these problems do not occur if the appropriate energy scale H_{eff} is used to calculate the coefficient of performance instead of using $C_{\mathrm{P,Betz}}$, cf. section IV.

The optimisation in approach (II) and (IV), which are both quasi-steady models, is only performed at an instant of the cycle, cf. (17) rather than over a full tidal circle as is necessary for obtaining an optimal array including its operation. In the case of Vennell this is addressed in a later study [27], and it is confirmed that a time-variant operation strategy $o(t)$ is needed for maximising the array power output. Only in approach (V) is an appropriate objective for optimising a tidal channel over a tidal cycle chosen.

The physical boundary conditions are usually given by the location of a tidal channel and are a function

TABLE I
ARRAY MODELS VIEWED FROM AN OPTIMISATION PERSPECTIVE.

			physical constraints			
		objective	boundary conditions	turbine model	array model	transient
(I)	Nishino & Willden [18]	$\max C_{\mathrm{P,Betz}}{}^1$	unaltered flow (Q =const.)	(i)	partial	steady
(II)	Vennell [22]	$\max(C_{\mathrm{P,Betz}} u_{\max}^3)$	head-driven, rough channel	(i)	full	quasi-steady
(III)	Vogel et al. [20]	$\max C_{\mathrm{P,Betz}}{}^1$	unaltered flow (Q =const.), Fr_1	(ii)	partial	steady
(IV)	Gupta & Young [26]	$\max C_{\mathrm{P,Betz}}{}^0$	head-driven, rough channel	(ii)	partial	quasi-steady
(V)	Bonar et al. [7]	$\max \int_0^T C_{\mathrm{P,Betz}}{}^0$	head-driven, rough channel	(ii)	partial	transient

(i) is the turbine model of Garret & Cummins [23], (ii) the model from Houlsby et al. [24] or Whealen et al. [17].
^{0} P_{avail} is calculated with the unaltered flow speed u_0 of the channel
^{1} P_{avail} is calculated with the flow speed u_1 far in front of the array

of time. The authors of model (I) and (II) falsely assume that the flow in the channel is unaltered by the array. This is only valid for low blockage ratios $\sigma_{\mathrm{A}} \ll 1,\ \sigma_l \ll 1$ and low turbine heads $H_{\mathrm{T}} \to 0$ which makes this boundary condition physically implausible, especially for larger arrays. In the other models a head-driven flow through a rough channel is applied which respects the influence of the added drag due to the array operation and its design.

The turbine model, which acts as a base for the presented array models is an essential physical constraint. The array models (I) & (II) are both based on the turbine model (i) of Garret & Cummins [23]. Hence both models can not be used for high turbine heads or highly blocked channels, where the free surface influence is not negligible.

The array models (III)–(V) are based on a turbine model proposed by Houlsby et al. [24] and by Whelan et al. [17]. This model includes free surface influences. A major disadvantage of this turbine model is that it becomes physically incorrect, i.e. wrong for relevant blockage ratios or high Froude numbers. The reason for this is, that the free surface influence on the deformation of the turbine stream tube is not considered. Pelz et al. [5] showed that the coefficient of performance and the volumetric efficiency are overestimated for high turbine heads, high blockages and high turbine heads and the result that the equations can not be solved for higher turbine heads, cf. Fig. 7. The predictions even become inconsistent with the energy and continuity equation for high turbine heads. For array model (III) this is also reported for high local blockages by the authors themselves. Hence models (III)–(V) are only valid if the blockage and Froude number are low.

The roughness of the seabed is included in model (II), (IV) and (V) which is important in the case of long tidal channels as shown by the dimensional analysis in section III. This is straight forward in the case of an array spanning the full channel width combined with turbine model (i), cf. array model (III): the main flow velocity u_1 is used to calculate the drag losses, which is in fact identical to the velocity after the turbine row, due to the neglected free surface deformation in the turbine model of Garret & Cummins [23]. This approach can not be used for partial arrays because of the array bypass flow, which has a length scale of order of the array length and a different velocity (usually higher than the array core flow). To calculate the drag losses properly the length of the array wake must be known. Generally, since the actuator disc theory is zero-dimensional, length scales can not be obtained from it and can only be modelled empirically, as done for array model (IV). Using the shallow water equations, model (V), brings the advantage of a two-dimensional model, where the bottom drag is considered in the modelling equations themselves.

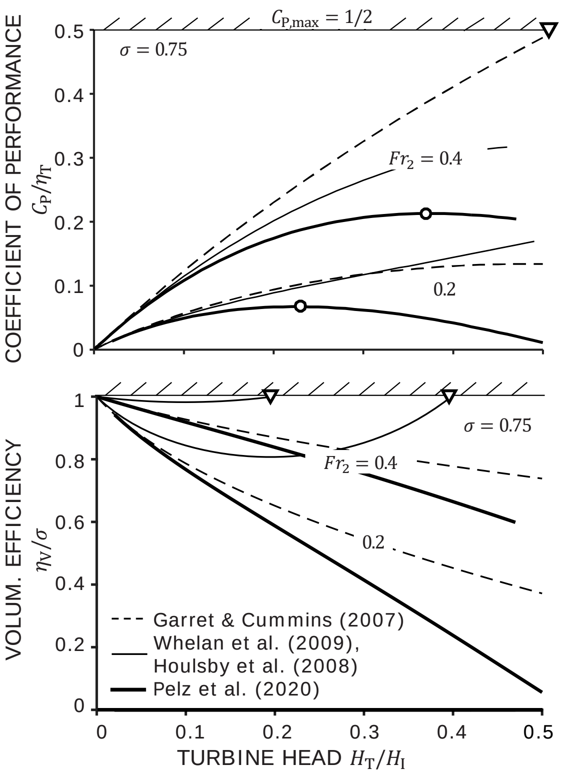


Fig. 7. Coefficient of performance (top) and specific volumetric efficiency (lower graph) versus turbine head $H_{\mathrm{T}}/H_{\mathrm{I}}$ for a single turbine with blockage of $\sigma = 0.75$. The bold lines represent turbine model (iii), the thin lines model (ii) and the dashed lines turbine model (i). The conflict with the energy (top) and the continuity equation (lower graph) are marked with a triangle (adapted from [5]).

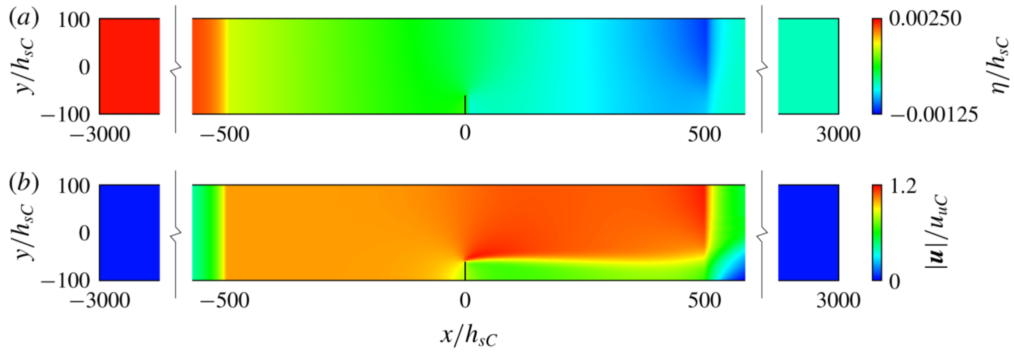


Fig. 8. Solution of the shallow water equations, calculated by Bonar et al. The unsteady elevation rise (a) and velocity drop in (b) indicate the presence of a hydraulic jump (from [7]).

The models discussed so far deal with generic channels in terms of a dimensionless model formulation, whereas Bonar et al. [7] investigate a specific tidal channel of length $L = 20\,\text{km}$, undisturbed water depth $h_0 = 20\,\text{m}$, and cycle time $T = 12.4\,\text{h}$, by solving the shallow water equations for this site.

The perturbation parameter ε, that was introduced in Sec. II, is also part of a dimensionless formulation of the shallow water equation. This becomes obvious by transforming the shallow water equations to a dimensionless form (this was not done by Bonar et al. [7]):

$$\underbrace{\varepsilon\frac{\partial h_+}{\partial t_+}}_{\sim\varepsilon} + \underbrace{\nabla_+\vec{q}_+}_{\sim 1} = 0. \tag{18}$$

With the dimensionless variables marked by the subscript $+$: $t := t_+T$, $x := x_+L, y := y_+L$, $u := u_+\sqrt{gh_0}$, $v := v_+\sqrt{gh_0}$, $h := h_+h_0, \eta := \eta_+h_0$.
$\vec{q}_+ = h_+\vec{U}_+$ is the volume flow vector with the depth averaged velocity vector $\vec{U}_+ = u_+\vec{e}_x + v_+\vec{e}_y$.
The momentum equations in dimensionless form read,

$$\underbrace{\varepsilon\frac{\partial \vec{q}_+}{\partial t_+}}_{\sim 10^{-2}} + \underbrace{\left[\nabla_+\cdot\mathbf{A} + \frac{1}{2}\nabla_+(h_+^2 - z_+^2)\right]}_{\sim 1} \tag{19}$$

$$= \underbrace{\eta_+\nabla_+z_+}_{\sim 1} + \underbrace{\frac{L}{h_0}c_f\vec{U}_+|\vec{U}_+|}_{\sim 1..10^{-1}} + \underbrace{\varepsilon\left(\frac{h_0}{L}\right)^2\frac{T\nu_t}{h_0^2}\nabla_+\cdot\nabla_+\vec{q}_+}_{\sim 10^{-2}..10^{-5}}.$$

For the tensor $\mathbf{A} = h_+\vec{U}_+\otimes\vec{U}_+$ the dyadic product of the velocity vector is used. η is the free surface elevation above the still water surface z_s, $h := \eta + z_s$ and $z_+ := z_sh_0$.

From a mathematical point of view, the transient terms $\partial/\partial t$ may be treated as a perturbation of the quasi-stationary solution of the problem. The perturbation parameter of this channel is $\varepsilon \approx 3.2\times10^{-2}$. Thus the use of a transient solver is unnecessary because quasi-steady conditions apply.

Fig. 8 shows one result obtained by Bonar et al. at the downstream position $x/h_{sC} \approx 500$ a jump (discontinuity) in both surface elevation η and flow velocity u is visible. This indicates the presences of a hydraulic jump. The necessary condition for a hydraulic jump is overcritical flow $Fr > 1$ which is usually not the case in a tidal channel. It is questionable if the findings of this work are valid for a general array in a tidal channel.

VI. Conclusion

Based on fundamental fluid mechanics and by using a dimensional analysis it was clarified that most tidal channels are quasi-stationary. This is very important because it results in a reduction of complexity since time is only an implicit parameter. We have shown that the optimisation of a tidal array can be treated as a two-stage optimisation problem. For this optimisation, the correct objective and boundary conditions have to be chosen. Using the coefficient of performance introduced by Betz as the objective is not appropriated because it uses an unsuitable energy reference and can therefore result in arrays falsely deemed 'optimal'. Furthermore, the optimisation must be done for a whole tidal cycle in order to call an array 'optimal' and not only at one point in time. Finally, the boundary conditions, especially the turbine model, need to be set correctly. All array models that were discussed here can not be used for high blockages ($\sigma > 0.25$) and at the same time high Froude numbers $Fr > 0.1$ and turbine heads due to the underlying turbine models (i) and (ii).

Only turbine model (iii) [5] is suitable for modelling high blockage arrays and the same time high turbine heads $\overline{H}_\text{T}$ or high Froude numbers. The extension of this model to arrays of tidal turbines is subject to future research.

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